General Linear Group

It is denoted by $GL_n(\mathbb{F})$ or $GL(n, \mathbb{F})$ and $n \in \mathbb{N}$ define as :-

$$GL_n(\mathbb{F}) = \{ A = [a_{ij}]_{n\times n} : |A| \neq 0, \ a_{ij} \in \mathbb{F} \}$$

here $A = [a_{ij}]_{n\times n} \rightarrow$ represent matrix of order $n \times n$.

$|A| = |[a_{ij}]_{n\times n}| \rightarrow$ represent determinent.

$\mathbb{F} \rightarrow$ represent field.

Explanation about matrix.

$[\ \ ]_{1\times 1}$ $\quad$ $[\ \ ]_{2\times 2}$ $\quad$ $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times 3}$

$[a_{11}]_{1\times 1}$ $\quad$ $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{2\times 2}$

corr to col

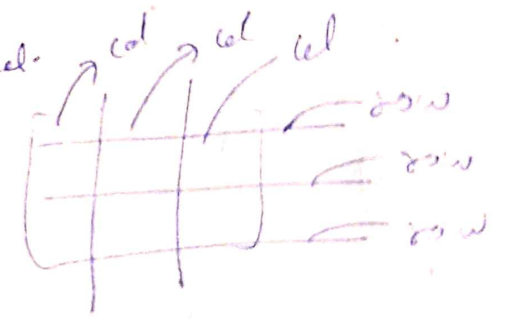
corr to row

corr to col.

corr. row

corr row

corr to row



a_{ij} → rep. row, rep. col.

QN Show that $GL_n(\mathbb{F})$ is group w.r.to Matrix Multiplication.

Proof $GL_n(\mathbb{F}) = \{ A = [a_{ij}]_{n\times n} : |A| \neq 0, \ a_{ij} \in \mathbb{F} \}$

(1) let $A \in GL_n(\mathbb{F}) \Rightarrow |A| \neq 0$,

$B \in GL_n(\mathbb{F}) \Rightarrow |B| \neq 0$

s.t. $|A \cdot B| = |A| \cdot |B| \neq 0$

$\Rightarrow$ Closure Property hold.

(ii) $A\cdot(B\cdot C) = ABC = (A\cdot B)\cdot C \ \forall\ A, B, C \in GL_n(\mathbb{F})$.

(iii) $I = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots & & & 1 \end{bmatrix}_{n\times n} \in GL_n(\mathbb{F})$ because $|I| = 1 \neq 0$

identity Matrix

s.t. $A\cdot I = I\cdot A = A \ \forall\ A \in GL_n(\mathbb{F})$.

(iv) Let $A \in GL_n(\mathbb{F})$ then $|A| \neq 0$

then $A^{-1} = \dfrac{\text{adj} A}{|A|}$ exist.

$\Rightarrow |A^{-1}|\,|A| = I$

$\Rightarrow \exists$ inverse $\forall\ A \in GL_n(\mathbb{F})$.

$\Rightarrow GL_n(\mathbb{F})$ is a group w.r.to. Multiplication.

Q.N Show that $SL_n(\mathbb{F})$ is group w.r.to Matrix Multiplication

~~Sol~~ Proof:- $SL_n(\mathbb{F}) = \{A \in GL_n(\mathbb{F}) : |A| = 1\}$

where $A = [a_{ij}]_{n\times n} \rightarrow$ matrix of order n.

(i) let $A \in SL_n(\mathbb{F})$ then $|A| = 1$

and $B \in SL_n(\mathbb{F})$ then $|B| = 1$

s.t. $|A\cdot B| = |A|\cdot|B| = 1\cdot 1 = 1$

$\Rightarrow A\cdot B \in SL_n(\mathbb{F})$.

Closure Property hold.

(ii) $A\cdot(B\cdot C) = ABC = (A\cdot B)\cdot C \ \forall\ A, B, C \in SL_n(\mathbb{F})$.

(iii) $I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}_{n\times n} \in SL_n(\mathbb{F})$ because $|I| = 1$

then $A\cdot I = I\cdot A = A \ \forall\ A \in SL_n(\mathbb{F})$.

(iv) let $A \in SL_n(\mathbb{F})$ then $|A| = 1 \neq 0$.

then $A^{-1} = \frac{adj\ A}{|A|}$ exist s.t.

$$|A^{-1}|\ |A| = I$$

$\Rightarrow \exists$ inverse $\forall\ A \in SL_n(\mathbb{F})$.

$\Rightarrow SL_n(\mathbb{F})$ is a group

Q Show that D_n is group.

D_n (Dihedral group).

Proof :- $D_n = \{ x^i y^j\ ;\ x^2 = e,\ y^n = e,\ xy = y^{-1}x \}$

$i = 0, 1,\ j = 0, 1, 2, \dots n-1$.

$x \longrightarrow$ Reflection

$y \longrightarrow$ Rotation.

(i) let $a \in D_n$ then a is Rotation or Reflection and $b \in D_n$ then b is Rotation or Reflection. s.t. $a \cdot b$ is Rotation or Reflection.

$\Rightarrow a \circ b \in D_n$. (Closure Property hold)

(ii) $a \circ (b \circ c) = (a \circ b) \circ c\ \forall\ a, b, c \in D_n$

(mapping composition satisfied)

Associative Property held.

(iii) $e \in D_n$ s.t. $a \cdot e = e \cdot a = a\ \forall\ a \in D_n$.

Existence of identity.

(iv) let $a \in D_n$ then a is Rotation or Reflection.

Case I ;- If a is Reflection then $a = x$.

s.t. $x^2 = e$

$\Rightarrow \quad a^2 = e$

$\Rightarrow \quad a \cdot a = e$

$\Rightarrow \quad a = \frac{e}{a} = e a^{-1}$

$\Rightarrow \quad a = a^{-1}$.

Case II :- If a is Rotation then

$a = y^r, \quad 1 \le r \le n$.

then $a^{-1} = (y^r)^{-1} = y^{n-r}$.

$y^n = e$.

$y^{n-r} \cdot y^r = e$

$\Rightarrow y^{n-r} = (y^r)^{-1}$

$\Rightarrow$ D_n is a group under mapping composition.

<u>Symmetric Group</u> :-

set of all one-one onto mapping from set containing n elements to itself from a group w.r.to mapping composition.

It is denoted by S_n and $O(S_n) = n!$

<u>QN</u> Show that S_n is group w.r.to mapping composition.

<u>Soln</u>:- (1) let $f \in S_n$ then f is one-one and onto mapping.

$g \in S_n$ then g is one-one and onto mapping

s.t. $f \circ g$ is one-one and onto.

then $f \circ g \in S_n$ Closure Property hold.

(ii) $f \circ (g \circ h) = f g h = (f \circ g) \circ h \quad \forall f, g, h \in S_n$.

Associative Property hold.

(iii) Identity mapping I is one-one and onto mapping then $I \in S_n$ s.t.

$$f \circ I = I \circ f = f \quad \forall \ f \in S_n.$$

(iv) let $f \in S_n$ ~~so~~ then f is one-one and onto.

$\Rightarrow$ f^{-1} exist.

$\Rightarrow$ f^{-1} is also one-one and onto.

$\Rightarrow$ $f^{-1} \in S_n$ s.t. $f \circ f^{-1} = f^{-1} \circ f = I$.

$\Rightarrow$ S_n is a group.